NOTES ON THE RICCATI EQUATION

Eugene S. KRYACHKO

Bogoliubov Institute for Theoretical Physics, Kiev, 03143 Ukraine and Department of Chemistry, Bat. B6c, University of Liège Sart-Tilman, B-4000 Liège 1, Belgium; e-mail: eugene.kryachko@ulg.ac.be

Received March 16, 2005
Accepted May 20, 2005

The French fictional detective Arsène Lupin, arch rival of Sherlock Holmes, said: "Il faut commencer à raisonner par le bon bout".
S. M. Ulam, Adventure of a Mathematician
(University of California Press, Berkeley 1991).

Faithfully dedicated to Joe Paldus on the occasion of his 70th birthday.

The relationship between the Riccati and Schrödinger equations is discussed. It is shown that the transformation converting the Riccati equation into its normal form is expressed in terms of the roots of its algebraic part treated as a second-order polynomial. Together with the well-known Riccati transformation, a new transformation which also links the Riccati equation to the second-order linear differential equation is introduced. The latter is actually the Riccati transformation applied to an "inverse" Riccati equation. Two specific forms of the Riccati equation admitting the explicit particular rational solutions are obtained.
Keywords: Schrödinger equation; Riccati transformation; Differential equations; Quantum chemistry.

The Riccati equation[1,+] is a first-order nonlinear differential equation (for more details, see e.g., refs.[2–10])

$$y'(x) = a(x)y^2(x) + b(x)y(x) + c(x)\,, \quad ac \neq 0 \qquad (1)$$

+ D'Alembert apparently was the first who named this equation in 1763 as "Riccati's equation".

doi:10.1135/cccc20050941

where $' \equiv \mathrm{d}/\mathrm{d}x$. The Riccati differential equation is widely used in many areas of quantum mechanics (see refs.[11–13] for current works), e.g., in the WKB approximation[14,++], in supersymmetric quantum mechanics[15], and particularly in quantum chemistry (see refs.[16–19] and references therein).

Let $y_0(x)$ be a particular solution of the Riccati equation (*1*). Defining then a new function $u(x)$ via

$$y(x) = y_0(x) + u(x) \tag{2}$$

and substituting (*2*) into (*1*), one obtains the first-order differential equation for $u(x)$

$$u'(x) = [b(x) + 2a(x)y_0(x)]u(x) + a(x)u^2(x)\ . \tag{3}$$

Equation (*3*) is the Bernoulli first-order differential equation[4]. Therefore, for a given particular solution of the Riccati equation, the latter converts to the Bernoulli equation which is solvable[4].

To find the general solution of (*3*), let us define $v = u^{-1}$. Equation (*3*) is then transformed into the following first-order linear differential equation for v

$$v' + (b + 2ay_0)v = -a\ . \tag{4}$$

The general solution of (*4*) is straightforward, viz.

$$v(x) = \frac{1}{A(x,x_0)}[v(x_0) - \int_{x_0}^{x} a(x_1)A(x_1,x_0)\,\mathrm{d}x_1] \tag{5}$$

where $A(x,x_0) = \exp\{\int_{x_0}^{x}[b(x_1) + 2a(x_1)y_0(x_1)]\mathrm{d}x_1\}$. Therefore, the general solution of the Riccati equation can be represented in terms of a linear fraction

++It seems that Young[14a] first derived the Riccati equation from the Schrödinger one (see Eq. (*7*) in his work[14a]).

$$y(x) = \frac{CA_1(x) + B_1(x)}{-CA_2(x) + B_2(x)} \tag{6}$$

where $C = y(x_0)$, $A_1(x) = A(x,x_0) - y_0(x)D(x,x_0)$, $B_1(x) = y_0(x) - A_1(x)y_0(x_0)$, $A_2(x) = D(x,x_0)$, $B_2(x) = 1 + y_0(x_0)A_2(x)$, and $D(x,x_0) = \int_{x_0}^{x} a(x_1)A(x_1,x_0)\,\mathrm{d}x_1$.

NORMAL FORM OF RICCATI EQUATION

Return to the Riccati equation (*1*) and apply the transformation $y(x) = z(x)/a(x)$. One obtains the canonical form of the Riccati equation for $z(x)$ [10]

$$z' = z^2 + \left(\frac{a'}{a} + b\right)z + ac \tag{7}$$

where the function $a(x)$ in (*1*) is converted to 1. Equation (*7*) can be rewritten as

$$z' = (z - \alpha_+)(z - \alpha_-) \tag{8}$$

where α_+ and α_- are the roots of the following second-order algebraic polynomial in z

$$z^2 + \left(b + \frac{a'}{a}\right)z + ac = 0\,. \tag{9}$$

Introducing $\beta_1 = (\alpha_+ + \alpha_-)/2$ and $\beta_2 = \alpha_+ - \alpha_-$ and substituting them into (*8*) yields

$$z' = (z - \beta_1)^2 - \frac{1}{4}\beta_2^2 \tag{10}$$

or

$$u' = u^2 + \left(\beta_1' - \frac{1}{4}\beta_2^2\right) \tag{11}$$

in terms of $u(x) = z(x) - \beta_1(x)$. The form (*11*), precisely

$$u'(x) = u^2(x) + C(x) \tag{12}$$

is, by definition, the normal form of the Riccati differential equation (*1*)[5–7] (see also refs.[20,21]). Using the expressions for the roots of Eq. (*9*), one can explicitly represent $C(x)$ as

$$C = -\frac{1}{2}\left(b + \frac{a'}{a}\right)' - \frac{1}{4}\left(b + \frac{a'}{a}\right)^2 + ac \tag{13}$$

which naturally coincides with the well-known formula for C (refs.[20,21]) (see also Eq. (*2*) in ref.[22]), though its representation (*13*) is more compact and derived in a different way.

Consider Eq. (*11*) and substitute $u = R \sin \alpha$ and $C = R \cos \alpha$ therein. One readily derives the equation for $z = 2\alpha$

$$z' = -\frac{C'}{C} \sin z + 2C \tag{14}$$

which, on one hand, coincides with Eq. (I.79) of Kamke[4] and on the other, resembles the Prüfer equation $\theta' = \cos^2 \theta + C \sin^2 \theta$ (see, e.g., ref.[23]).

RICCATI AND SCHRÖDINGER EQUATIONS

In 1760, Euler proved that the Riccati first-order nonlinear differential equation (*1*) can be equivalently reduced to a second-order linear homogeneous differential equation (see, e.g., refs.[2–4]). The Schrödinger equation is of the latter type, that is, in other words, there exists the correspondence between Riccati and Schrödinger equations (see, e.g., ref.[11] for a current work and references therein). To demonstrate this correspondence, let us define the following transformation

$$y(x) = \beta(x)\frac{\psi'(x)}{\psi(x)} + \alpha(x) \tag{15}$$

and substitute it, together with the corresponding expression for $y'(x)$, into Eq. (*1*). As a result, one obtains

$$\frac{\psi''}{\psi} - \left(\frac{\psi'}{\psi}\right)^2 (1 + a\beta) + \frac{\psi'}{\psi}\left(\frac{\beta'}{\beta} - 2a\alpha - b\right) + \frac{1}{\beta}(\alpha' - a\alpha^2 - b\alpha - c) = 0 \ . \qquad (16)$$

There are a number of choices and alternatives to deal with Eq. (*16*):
(i) $\alpha = 0$. Equation (*16*) then simplifies to

$$\frac{\psi''}{\psi} - \left(\frac{\psi'}{\psi}\right)^2 (1 + a\beta) + \frac{\psi'}{\psi}\left(\frac{\beta'}{\beta} - b\right) - c\frac{1}{\beta} = 0 \ . \qquad (17)$$

To remove the term $(\psi'/\psi)^2$ in (*17*), one has to put $\alpha\beta + 1 = 0$ that yields

$$a(x)\psi''(x) - [a'(x) + a(x)b(x)]\psi'(x) + a^2(x)c(x)\psi(x) = 0 \ . \qquad (18)$$

Equation (*18*) is a well-known second-order linear homogeneous differential equation[6,7]. The transformation $y = -\psi'/a\psi$ is called the Riccati transformation (see also the work by Liouville[24]). Note also that the Riccati transformation $u = -\psi'/\psi$ converts the normal Riccati equation (*12*) into

$$\psi''(x) + C(x)\psi(x) = 0 \ ; \qquad (19)$$

(ii) $\alpha \neq 0$ and $\beta = -1/a$. $\alpha = -b/2a$ yields the second-order differential equation

$$\psi'' - \frac{a'}{a}\psi' + \left(ac - \frac{b^2}{4} - \frac{a'b}{2a} + b'\right)\psi = 0 \ . \qquad (20)$$

This transformation $y = -\psi'/(a\psi) - b/(2a)$ has recently been introduced in ref.[12] Finally notice that $\alpha = -a'/(2a^2) - b/(2a)$ gives rise to the second-order linear differential equation

$$\psi'' + \left(ac - \frac{b^2}{4} - \frac{a'b}{2a} + \frac{b'}{2} - \frac{3(a')^2}{4a^2} + \frac{a''}{2a}\right)\psi = 0 \qquad (21)$$

which does not include the term with the first derivative ψ'.

There exists another transformation that also links the Riccati equation (*1*) to a second-order linear differential equation. Let us introduce a new function $\phi(x)$ such that

$$y(x) = \beta(x)\frac{\phi(x)}{\phi'(x)} + \alpha(x) . \qquad (22)$$

Substituting (*22*), with $\alpha(x) = 0$, together with the corresponding expression for $y'(x)$, into Eq. (*1*), results in

$$\frac{\phi\phi''}{(\phi')^2} + \alpha\beta\left(\frac{\phi}{\phi'}\right)^2 + \frac{\phi}{\phi'}\left(b - \frac{\beta'}{\beta}\right) + \left(\frac{c}{\beta} - 1\right) = 0 . \qquad (23)$$

If $\beta(x) = c(x)$, one readily derives the following second-order linear differential equation

$$c(x)\phi''(x) - [c'(x) + c(x)b(x)]\phi'(x) + a(x)c^2(x)\phi(x) = 0 . \qquad (24)$$

Therefore, there actually exist two different second-order linear homogeneous differential equations (*18*) and (*24*) linked, respectively, to the Riccati equation by means of the Riccati transformation $y = -\psi'/a\psi$ and by a new one, obtained from the ansatz, $y = c\phi/\phi'$. Obviously, Eq. (*24*) is more suitable compared to Eq. (*18*) at the nodes of $\psi(x)$ although the latter usually corresponds, as for instance in the supersymmetric approach, to the ground eigenwavefunction either of Eq. (*19*) or (*18*). Nevertheless, both these transformations are formally equivalent that can be readily demonstrated by applying the inverse transformation

$$y(x) = \frac{1}{z(x)} \qquad (25)$$

to Eq. (*1*). This yields the equation

$$z'(x) = -c(x)z^2(x) - b(x)z(x) - a(x) \qquad (26)$$

which is also of the Riccati type.

CLASSIFICATION OF THE RICCATI EQUATION: AN ATTEMPT

The well-known theorem in the theory of the Riccati differential equation[8,9] (see also ref.[25]) states that if $f(x)$, $g(x)$, $\phi(x)$ and $\psi(x)$ are arbitrary differentiable functions

$$y(x) = \frac{C f(x) + g(x)}{C \phi(x) + \psi(x)} \tag{27}$$

where C is a constant, obeys the Riccati equation. This particularly resembles the aforementioned Eq. (*6*).

One may suggest that Eq. (*27*) chosen in a specific analytical form is a particular solution $y_0(x)$ of the corresponding Riccati equation. Using further the transformation $y(x) = y_0(x) + 1/u(x)$, proposed by Euler in 1760 (see, e.g., refs.[6,12]) and which is analogous to that in introduction, one converts a given Riccati equation to the explicitly solvable Bernoulli equation for $u(x)$ (ref.[4]) and, hence, obtains the general solution of a given Riccati equation. Furthermore, applying either the Riccati transformation $y = -\psi'/a\psi$ or $y = c\phi/\phi'$ to the last Riccati equation, one transforms it to a particular second-order linear differential equation and hence obtains its general solution as well. In order to demonstrate the above suggestion, let us consider the following cases:

(i) $f = g$ and $\phi = \psi$. It merely leads to the Bernoulli differential equation

$$y' = \frac{\phi\psi' - \phi'\psi}{\psi} y^2 - \frac{\psi'}{\psi} y \; ; \tag{28}$$

(ii)

$$y(x) = \frac{C f + 1}{C \phi + \psi} \, . \tag{29}$$

Assuming that $f\psi - \phi = 1$, Eq. (*29*) results in the Riccati equation with the following coefficients

$$c = -f' , \quad b = 2f'\psi , \quad a = \frac{bc' - b'c + 2b^2 c}{4c^2} \tag{30}$$

whose particular solution casts explicitly as

$$y(x) = \frac{1 - C\int^{x} c(x)\,\mathrm{d}x}{\dfrac{b(x)}{2c(x)} - C\left[1 + \dfrac{b(x)}{2c(x)}\int^{x} c(x)\,\mathrm{d}x\right]}\,; \tag{31}$$

(iii) Define the following two forms of Eq. (*27*):

$$y_1(x) = \frac{C\sin\alpha(x) + \cos\alpha(x)}{C\cos\alpha(x) + \sin\alpha(x)}$$

$$y_2(x) = R(x)\frac{C\sin\alpha(x) + \cos\alpha(x)}{C\cos\alpha(x) + \sin\alpha(x)}\,. \tag{32}$$

After some algebra, one obtains that both $y_1(x)$ and $y_2(x)$ satisfy Riccati-type equations

$$y_1' = -\frac{\alpha'}{\cos 2\alpha}y_1^2(x) - \frac{\alpha'}{\cos 2\alpha}$$

$$y_2' = -\frac{\alpha'}{R\cos 2\alpha}y_1^2(x) - \frac{R'}{R}y_1 - \frac{R\alpha'}{\cos 2\alpha}\,. \tag{33}$$

A comparison of the former one with Eq. (*1*) yields $a(x) = c(x) = -\alpha'(x)/\cos 2\alpha(x)$ and $b(x) = 0$. For a given $a(x) = c(x)$ in the Riccati equation (*1*) one easily derives

$$\alpha(x) = -\frac{\pi}{4} + \arctan\left\{\tan\left(\frac{\pi}{4} + \alpha(x_0)\right)\exp\left[-4\int_{x_0}^{x} a(x)\,\mathrm{d}x\right]\right\}. \tag{34}$$

The Riccati transformation converts the first Riccati equation in Eq. (*33*) into the following second-order linear differential one

$$\psi''(x) - \frac{c'(x)}{c(x)}\psi'(x) + c^2(x)\psi(x) = 0\,. \tag{35}$$

By analogy, the last equation in (*33*) determines $a(x) = -\alpha'(x)/R(x)\cos 2\alpha(x)$, $c(x) = -R(x)\alpha'(x)/\cos 2\alpha(x)$, and $b(x) = R'(x)/R(x)$. Therefore, one may explicitly determine $\alpha(x)$ and $R(x)$ of $y_2(x)$ given by Eq. (*32*)

$$R(x) = R(x_0)\exp\left[-\int_{x_0}^{x} b(x)\,\mathrm{d}x\right]$$

$$\alpha(x) = -\frac{\pi}{4} + \arctan\left\{\tan\left(\frac{\pi}{4} + \alpha(x_0)\right)\exp\left[-4\int_{x_0}^{x} a(x)R(x)\,dx\right]\right\}. \qquad (36)$$

since $c(x) = a(x)R^2(x)$. This Riccati equation converts to a more complicated second-order linear differential equation determined by the coefficients $a(x)$, $b(x)$ and $c(x)$.

CONCLUSIONS

To conclude, we have thus demonstrated the existence of another transformation that converts the Riccati equation into a second-order linear differential equation. This is actually the well-known Riccati transformation for an "inverse" Riccati equation (*26*). Further, using the rational representation (*27*), satisfying the Riccati equation, we have found two particular solutions of the associated Riccati equations that can be used to determine general solutions of the corresponding second-order linear differential equations, either derived via the Riccati or the inverse transformation.

I gratefully thank Erkki Brändas for valuable comments and suggestions, Francoise Remacle for warm hospitality, José Manuel Garcìa de la Vega and Eric Fraga for kind offering me their book with Serafin Fraga on the Schrödinger and Riccati equations[18d]*, Brian Burrows for providing a reprint of his recent paper on the Riccati equation*[19a]*, and the F.R.F.C. 2.4562.03F (Belgium) for fellowship.*

REFERENCES

1. Riccati J.: *Actorum Eruditorum quae Lipsiae Publicantur* **1724**, Suppl. *8*, 66.
2. Watson G. N.: *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, London 1958.
3. Ince E. L.: *Ordinary Differential Equations*. Dover, New York 1956.
4. Kamke E.: *Differentialgleichungen: Lösungsmethoden und Lösungen*. Becker & Erler Kom.-Ges., Leipzig 1943.
5. Reid W. T.: *Riccati Differential Equations*. Academic Press, New York 1972.
6. Boyce W. E., DiPrima R. C.: *Elementary Differential Equations and Boundary Value Problem*. Wiley, New York 1986.
7. Zwillinger D.: *Handbook of Differential Equations*. Academic Press, Boston 1989.
8. Stepanov V. V.: *A Course of Differential Equations*. GITTL, Moscow 1953.
9. Zelikin M. I.: *Homogeneous Spaces and Riccati Equation in Variational Calculus*. Factorial, Moscow 1998.
10. a) Polyanin A. D., Zaitsev V. F.: *Handbook of Exact Solutions for Ordinary Differential Equations*. CRC Press, Boca Raton 1995; b) Zaitsev V. F., Polyanin A. D.: *Handbook of Ordinary Differential Equations*. FizMatLit, Moscow 2001.
11. Haley S. B.: *Am. J. Phys.* **1997**, *65*, 237.

12. Nowakowski M., Rosu H. C.: *Phys. Rev. E: Stat. Phys., Plasmas, Fluids, Relat. Interdiscip. Top.* **2002**, *65*, 047602.
13. Krivec R., Mandelzweig V. B.: *Comput. Phys. Commun.* **2003**, *152*, 165.
14. a) Young L. A.: *Phys. Rev.* **1931**, *38*, 1612; b) Brändas E., Hehenberger M.: *Lect. Notes Math.* **1974**, *415*, 316; c) Herman Z. S., Brändas E.: *Mol. Phys.* **1975**, *29*, 1545; d) Shankar R.: *Principles of Quantum Mechanics*, p. 240. Plenum, New York 1980; for recent work see e) Ma Z.-Q., Xu B.-W.: *Europhys. Lett.* **2005**, *69*, 685.
15. Cooper F., Khare A., Sukhatme U.: *Supersymmetry in Quantum Mechanics*. World Scientific, Singapore 2001; and references therein.
16. a) Bessis N., Bessis G.: *Phys. Rev. A: At., Mol., Opt. Phys.* **1996**, *53*, 1330; b) Bessis N., Bessis G.: *J. Chem. Phys.* **1995**, *103*, 3006.
17. a) Montemayor R., Salem L. D.: *Phys. Rev. A: At., Mol., Opt. Phys.* **1991**, *44*, 7037; b) Fernandez F. M.: *Phys. Lett. A* **1994**, *194*, 343; c) Berrondo M., Recamier J.: *Int. J. Quantum Chem.* **1997**, *62*, 239; d) Morales J., Pena J. J., Morales-Guzman J. D.: *Theor. Chem. Acc.* **2000**, *104*, 179; e) Chakrabarti B., Das T. K.: *Phys. Lett. A* **2001**, *285*, 11.
18. a) Fraga S., Fraga E. S.: *J. Mol. Struct. (THEOCHEM)* **1998**, *426*, 1; b) Fraga S., Garcìa de la Vega J. M., Fraga E. S.: *Theor. Chem. Acc.* **2001**, *106*, 434; c) Fraga S., Garcìa de la Vega J. M., Fraga E. S.: *Can. J. Phys.* **2002**, *80*, 1053; d) Fraga S., Garcìa de la Vega J. M., Fraga E. S.: *The Schrödinger and Riccati Equations*. *Lect. Notes Chem.* **1999,** *70*.
19. a) Burrows B. L., Cohen M.: *Mol. Phys.* **2004**, *102*, 1165; b) Burrows B. L., Cohen M. in: *Fundamental World of Quantum Chemistry: A Tribute to the Memory of Per-Olov Löwdin* (E. J. Brändas and E. S. Kryachko, Eds), Vol. 3. Kluwer, Dordrecht 2004.
20. Bank S. B., Gundersen G. G., Laine I.: *Ann. Acad. Sci. A I Math.* **1981**, *6*, 369.
21. Zheng J. H.: *J. Math. Anal. Appl.* **1995**, *190*, 285.
22. Yuan W.: *J. Math. Anal. Appl.* **2003**, *277*, 367.
23. a) Prüfer H.: *Math. Ann.* **1926**, *95*, 409; b) Drukarev G. F.: *Zh. Eksp. Teor. Fiz.* **1949**, *19*, 247; c) Francetti S.: *Nuovo Cim.* **1957**, *6*, 601; see also d) Coddington E. A., Levinson N.: *Theory of Ordinary Differential Equations*. McGraw–Hill, New York 1955; e) Mielnik B., Reyes M. A.: *J. Phys. A: Math. Gen.* **1996**, *29*, 6009; f) Last Y., Simon B.: *J. Funct. Anal.* **1998**, *154*, 513; g) Schmidt K. M.: *J. Comput. Appl. Math.* **2004**, *171*, 393.
24. Liouville J.: *J. Math. Ser. I* **1841**, *VI*, 1.
25. Strelchenya V. M.: *J. Phys. A: Math. Gen.* **1991**, *24*, 4965.